SLOW NONISOTHERMAL FLOW OF A VISCOUS GAS BETWEEN TWO COAXIAL DISKS

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We consider the flow of a fluid between coaxial disks. An extensive literature deals with this problem; a review can be found in [1], for example. Detailed study has also been devoted to the study of flows in channels with permeable walls (with injection or evacuation) [2]. In both cases an exact solution of the Navier–Stokes equations can be found in the one-dimensional case, if one assumes that the fluid is incompressible.

Evidently an analogous one-dimensional solution in the case of a compressible gas cannot be found, in general. However if one considers a slowly flowing viscous gas, where the Mach number is small and therefore can be neglected, and the density variation is due to large temperature variations, then under certain conditions the one-dimensional nature of the solution can be retained. Due to the importance of this problem in applications (for example in chemical engineering and microelectronics), we obtain in the present paper a one-dimensional solution of the Navier–Stokes equations for the flow of a viscous, heat-conducting gas in the gap between two disks.

Let the lower disk rotate with angular velocity ω, while gas is injected with velocity v_{in} through the upper disk. We introduce a cylindrical coordinate system with the z-axis directed along the axis of the disks. Since the Mach number is assumed small, we can neglect the contribution of the kinetic energy to the total energy of the gas and do not take into account viscous dissipation in the energy equation. In addition, if the pressure variation in the channel is small, then in the equation of state we can neglect this variation and assume that the density of the gas depends only on its temperature. The concrete condition for the applicability of this assumption is discussed below.

With the above assumptions, we can represent the solution of the problem in the following form (the index 0 refers to parameters at z = 0)

$$v_r = r\frac{\nu_0}{h^2}F\left(\frac{z}{h}\right),\quad v_\varphi = \omega r G\left(\frac{z}{h}\right),\quad v_z = \frac{\nu_0}{h}H\left(\frac{z}{h}\right); \tag{1a}$$

$$p = -\rho_0\frac{\nu_0^2}{2h^4}ar^2 + \rho_0\frac{\nu_0^2}{h^2}P\left(\frac{z}{h}\right) + p_0\exp\left(-\frac{gz}{RT_0}\right),$$
$$T = T_0\Theta\left(\frac{z}{h}\right),\quad \rho = \rho_0\Theta^{-1}\left(\frac{z}{h}\right), \tag{1b}$$

where h is the channel thickness; ν is the kinematic viscosity; g is the acceleration of gravity; R is the universal gas constant; a is a parameter related to the injection velocity. The last term in (1b) takes into account the effect of gravity (along the z-axis) in the momentum equation.

Equation (1b) shows that the variation of the pressure in the equation of state will be small if

$$\frac{gh}{RT_0} \ll 1,\quad \rho_0\frac{\nu_0^2}{h^2} \ll p_0 \tag{2a}$$

and

$$r \ll \frac{2h^4}{\nu_0^2 a}\frac{p_0}{\rho_0}. \tag{2b}$$

Hence even for infinite disks the solution (1) is only valid in a finite region of r satisfying (2). On the other hand, despite the small variation of pressure, the first term in (1b)

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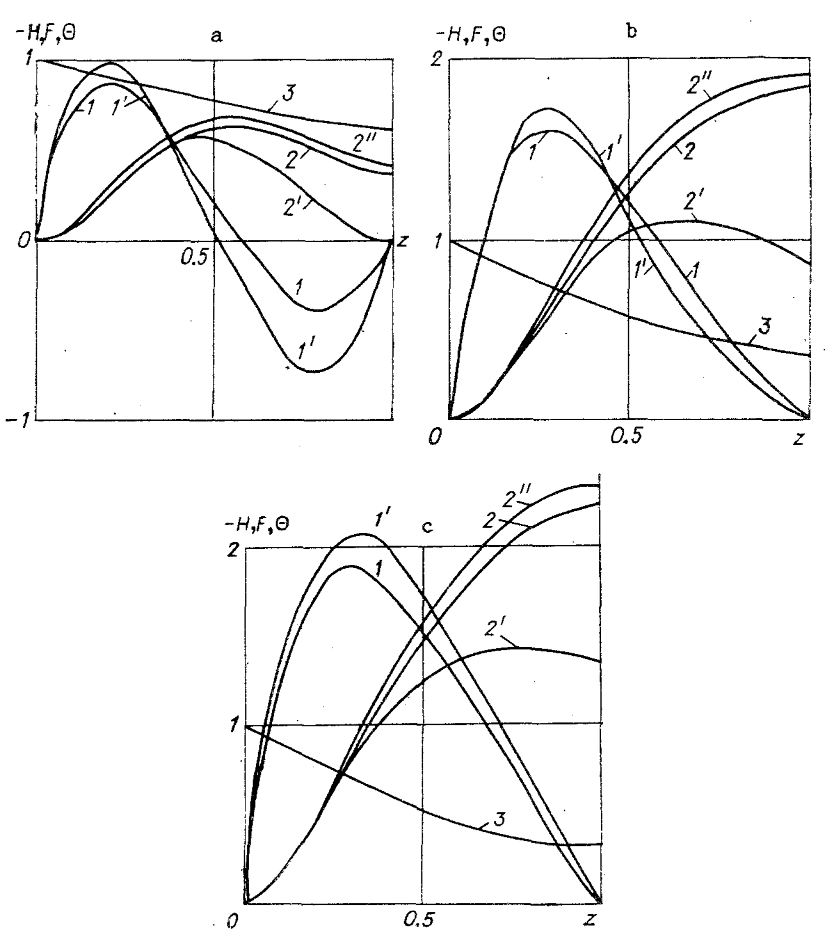

Fig. 1

(determining the radial pressure gradient) will significantly affect the nature of the flow in the channel.

Further, let the thermal conductivity λ and dynamic viscosity μ have the forms

$$\lambda = \lambda_0\Theta^n,\ \mu = \mu_0\Theta^m. \qquad (3)$$

Then, substituting (1) and (3) into the complete system of Navier–Stokes equations (which are not written out here), and using the above assumptions, the following system of ordinary differential equations is obtained:

$$H' = \Theta^{-1}\Theta' H - 2F, \qquad (4)$$

$$\begin{aligned} F'' &= \Theta^{-(m+1)}(F^2 - \mathrm{Ec}^{-2}G^2 + HF' - a\Theta) - m\Theta^{-1}\Theta' F'; \\ G'' &= \Theta^{-(m+1)}(2FG + HG') - m\Theta^{-1}\Theta' G', \\ \Theta'' &= \mathrm{Pr} H\Theta^{-(n+1)}\Theta' - n\Theta^{-1}\Theta'^2; \end{aligned} \qquad (5)$$

$$p' = -HH'\Theta^{-1} + \mathrm{Ga}\left(1 - \Theta^{-1}\right) + \frac{4}{3}\left(\Theta^m H'\right)' + \frac{2}{3}\Theta^m F' - \frac{4}{3}m\Theta^{m-1}\Theta' F, \qquad (6)$$

where Ec, Pr, and Ga are the Ecman, Prandtl, and Galilei numbers, defined as

$$\mathrm{Pr} = \frac{\rho_0 \nu_0 c_p}{\lambda_0}, \quad \mathrm{Ec} = \frac{\nu_0}{\omega h^2}, \quad \mathrm{Ga} = \frac{gh^3}{\nu_0^2}.$$

Equations (4) and (5) are coupled, while (6) determines the pressure. In the derivation of (6), the exponent in (1b) was expanded in a series and only the linear term was retained; this is justified, in view of (2).

For comparison, we give the analogous system of equations for an incompressible fluid, when the viscosity and thermal conductivity depend on temperature:

$$H' = -2F, \qquad (7)$$

$$F'' = \Theta^{-m}(F^2 - \mathrm{Ec}^{-2}G^2 + HF' - a) - m\Theta^{-1}\Theta'F'; \tag{8}$$
$$G'' = \Theta^{-m}(2FG + HG') - m\Theta^{-1}\Theta'G',\ \Theta'' = \Pr H\Theta^{-n}\Theta' - n\Theta^{-1}\Theta'^2;$$
$$p' = -HH' + \mathrm{Ga}(1 - \Theta^{-1}) + 2\Theta^m F' + 2(\Theta^m H')'. \tag{9}$$

The boundary conditions for the system (4) are:

$$F(0) = F(1) = 0,\ G(0) = 1,\ G(1) = 0,\ H(0) = 0,$$

and there is an additional condition on H(1), relating it to the given injection velocity: $H(1) = -hv_{in}/\nu_0$. This condition can be satisfied by properly choosing the parameter a for a fixed value of the Ecman number. Given a, on the other hand, one can find the injection velocity from the solution of the system.

The boundary conditions for the temperature are

$$T = T_0 \text{ for } z = 0,$$
$$-\lambda\frac{\partial T}{\partial z} = (-\rho v_z c_p + \alpha)(T - T_\infty) \tag{10}$$
$$\text{for } z = h,$$

where α is the coefficient of heat exchange with the surrounding medium. In the second boundary condition we specialize to the problem with injection and include the term $\rho v_z c_p$, which takes into account the discharge of heat due to heating of the injected gas, whose initial temperature is T_∞. Passing through the upper wall of the channel, the gas is heated and it enters the channel with a temperature equal to the temperature of the wall. Substituting (1) into (10), we obtain finally

$$\Theta|_{z=0} = 1,\ \Theta' - (\Pr\Theta^{-(n+1)}H(1) - \mathrm{Bi}\Theta^{-n})(\Theta - \Theta_\infty)|_{z=1} = 0, \tag{11}$$
$$\mathrm{Bi} = h\alpha/\lambda_0.$$

The above boundary-value problem was solved by a combination of the method of calibration and continuation of the solution in a parameter. The results of the calculation are shown in Fig. 1.

We note that for the system (7) through (9) the boundary condition for Θ has the form

$$\Theta' - \Theta^{-n}(\Pr H(1) - \mathrm{Bi})(\Theta - \Theta_\infty)|_{z=1} = 0. \tag{12}$$

Calculations were done for Ec = 0.1, Pr = 0.7, Bi = 1.0, m = n = 1/2; a = −18.8, 0, 5 [Fig. 1a through 1c, respectively, where curves 1 and 1' give F, curves 2, 2', 2" give H, curve 3 gives Θ; curves 1 and 2 were constructed with $\Theta_\infty = 1$; curves 1' and 2' with $\Theta_\infty = 0.3$; and curve 2" was obtained from the solution of the system (7) and (8) with $\Theta_\infty = 0.3$].

The case $\Theta_\infty = 1$ corresponds to the solution for an incompressible fluid, since in this case $\Theta = 1$ over the entire height of the channel and therefore the temperature distribution does not affect the velocity field. From Fig. 1 it is evident that the compressibility of the gas, caused by the nonuniformity of the temperature, significantly affects the velocity field. When $\Theta_\infty < 1$ the axial velocity H(z) decreases over the entire width of the channel and it becomes nonmonotonic even when $a > 0$, whereas the radial velocity F(z) decreases near the lower disk and increases near the upper disk. We emphasize that the difference between the solution for an incompressible fluid and that taking the compressibility of the gas into account increases if one takes into account the temperature dependence of λ and μ in keeping the density ρ constant.

LITERATURE CITED

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